Portfolio Insurance: determination of a dynamic CPPI multiple as function of state variables.

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December 2006, Preliminary version.

Abstract

This paper examines the CPPI method (*Constant Proportion Portfolio Insurance)* when the multiple is allowed to vary over time. A quantile approach is introduced, which can provide explicit values of the multiple as function of the past asset returns and volatility. These bounds can be statistically estimated from the behaviour of variations of rates of asset returns, using for example ARCH type models. We show how the multiple can be chosen to satisfy the guarantee condition, at a given level of probability and for particular market conditions.

JEL: C 10, G 11.

1 Introduction

The purpose of portfolio insurance is to give to the investor the ability to limit downside risk in bearish financial market, while allowing some participation in bullish markets. There exist several methods of portfolio insurance: OBPI (Option Based Portfolio Insurance), CPPI (Constant Proportion Portfolio Insurance), Stop-loss, ... (see for example Poncet and Portait (1997) for a review about these methods). Here, we are examine the CPPI method introduced by Black and Jones (1987) for equity instruments and Perold (1986, 1988) for fixed-income instruments (see also Black and Rouhani (1987), Roman, Kopprash and Hakanoglu (1989), Black and Perold (1992)). The CPPI method is based on a simplified strategy to allocate assets dynamically over time. Basically, two assets are involved: the riskless asset, B, with a constant interest rate r (usually Treasury bills or other liquid money market instruments) and the risky one, S (usually a market index).

Usually, the investor, with initial amount to invest V_0, wants to recover a fixed percentage α of his initial investment at a given date in the future T. To obtain a terminal portfolio value V_T greater than the insured amount αV_0, the portfolio manager keeps the portfolio value V_t above the floor $F_t = \alpha.V_0.e^{-r(T-t)}$ at any time t in the period $[0, T]$. For this purpose:

- The amount e_t invested in the risky asset is a fixed proportion m of the excess C_t of the portfolio value over the floor.
- The constant m is usually called *the multiple,* e_t *the exposure* and C_t *the cushion.*
- Since $C_t = V_t - F_t$, this insurance method consists in keeping C_t positive at any time t in the period.
- The remaining funds are invested in the riskless asset B_t.

Both the floor and the multiple are functions of the investor's risk tolerance. The higher the multiple, the more the investor will benefit from increases in stock prices. Nevertheless, the higher the multiple, the faster the portfolio will approach the floor when there is a sustained decrease in stock prices. As the cushion approaches zero, exposure approaches zero, too. In continuous-time, when asset dynamics have is no jump, this keeps portfolio value from falling below the floor. Nevertheless, during financial crises a very sharp drop in the market may occur before the manager has a chance to trade. This implies that m must not be too high (for example, if a fall of 10% occurs, m must not be greater than 10 in order to keep the cushion positive). Advantages of this strategy over other approaches to portfolio insurance are its simplicity and its flexibility (see for example De Vitry and Moulin (1994), Black and Rouhani (1987) and Boulier and Sikorav (1992)). Initial cushion, floor and tolerance can be chosen according to the own investor's objective (see for example Poncet and Portait (1997) and Prigent (2001)). Banks may bear market risks on the insured portfolios. In that case, banks can use, for example, stress testing since they may bear consequences of sudden large market decreases. For instance,in the case of the CPPI method, banks must, at least, provision the difference on their own capital if the value of the portfolio drops below the floor. Thus, one crucial question for the bank which promotes such funds is : what exposure to the risky asset or, equivalently, what level of the multiple to accept? On one hand, as portfolio expectation return is increasing with respect to the multiple, customers want the multiple as high as possible. On the other hand, due to market imperfections [1], portfolio managers must impose an upper bound on the multiple:

- First, if the portfolio manager anticipates that the maximal daily historical drop (e.g. -20 %) will happen during the period, he chooses $m \leq 5$ which leads to low return expectation. Alternatively, he may consider that the maximal daily drop during the management period will never be greater than a given value (e.g. -10%). A straightforward implication is to choose m according to this new extreme value (e.g. $m \leq 10$). Another possibility is to take account of the occurrence probabilities of extreme events in the risky asset returns.

- Second, he can adopt a quantile hedging strategy. In this case, he determines the multiple as high as possible but so that the portfolio value will always be above the floor at a given probability level (typically 99%).

The answer to this latter question has important practical implications. The determination of the multiple according to quantile conditions have been examined for instance in Prigent (2001) in the case of a Lévy process and in Bertrand and Prigent (2002), using extreme value theory. Hamidi, Jurczenko and Maillet (2006) also consider a modified quantile hedging strategy and shows how a conditional multiple can be determined.

In this paper, we also determine how the portfolio manager can choose the value of the multiple according to market fluctuations. This allows the introduction of a conditional multiple. To illustrate our approach, we consider several quantile conditions and introduce a quite general Garch type model. Our results prove that we can determine a conditional multiple as functions of state variables. These latter ones are the past stock logretruns and volatilities.

The paper is organized as follows. Section 2 presents the basic properties of the CPPI model. Section 3 introduces the modified CPPI method with a conditional multiple, based on various quantile conditions. In particular, upper bounds on the multiple are provided. Section 4 presents some simulations of our approach.

[1] For example, portfolio managers cannot actually rebalance portfolios in continuous time. Additionally, problems of asset liquidity may occur, especially during stock markets crashes.

2 The Model

2.1 *The financial market*

Consider two basic financial assets: a riskless asset, denoted by B, and a risky asset denoted by S (a financial stock or index price).

Changes in asset prices are supposed to occur at discrete times along a whole period $[0, T]$.

The riskless asset evolves according to a deterministic rate denoted by r.

The variations of the stock price S between two times t_k and $t_{k+1}$are defined by:

$$\Delta S_{t_{k+1}} = S_{t_{k+1}} - S_{t_k}.$$

Since we search an upper bound on the multiple m (see Proposition 1 in the following), we have to focus on the left hand side of the probability distribution of $\frac{\Delta S_{t_{k+1}}}{S_{t_k}}$. Thus, we introduce the notation :

$$X_{k+1} = -\frac{\Delta S_{t_{k+1}}}{S_{t_k}} = \frac{S_{t_k} - S_{t_{k+1}}}{S_{t_k}}$$

So X_k denotes the opposite of the relative jump of the risky asset at time t_k.

In fact, when we want to determine an upper bound on the multiple m, we have only to consider positive values of X.

Denote by M_n, the maximum of the finite sequence $(X_k\ )_{1 \leq k \leq n}$. We have:

$$M_n = Max(X_1, ..., X_n).$$

2.2 The CPPI portfolio

Denote by V_k the value of the portfolio at time t_k. As explained in the introduction, the CPPI method is based on the following portfolio insurance condition:

- There exists a deterministic floor F_k such that at any time t_k, the value V_k must be above the floor.
- The total amount e_k invested on the underlying asset S is equal to mC_k where the cushion C_k represents the difference $V_k - F_k$ between the portfolio value and the floor.
- The multiple m is a nonnegative constant.
- The remaining amount $(V_k - e_k)$ is invested on the riskless asset with a deterministic rate r_k for the period $[t_{k-1}, t_k]$.

The higher the multiple m, the higher the amount invested on the risky asset. Therefore, a speculative investor would choose high values for m. Nevertheless, in this case, his portfolio is riskier and as shown in what follows, his guarantee may no longer hold. Indeed, we easily deduce that the portfolio value is solution of

$$V_{k+1} = V_k - e_k X_{k+1} + (V_k - e_k) r_{k+1},$$

from which, we obtain the dynamics of the cushion :

$$C_{k+1} = C_k \left[1 - m\ X_{k+1} + (1-m) r_{k+1}\right]$$

Since, for all times t_k, the cushion must be positive, we get finally the condition : for all $k \leq n$,

$$-mX_k + (1-m) r_k \geq -1$$

In fact, since r_k is relatively small, the previous inequality yields to the following relation that gives an upper bound on the multiple:

Proposition 1 *The guarantee is satisfied at any time of the management period with a probability equal to* 1 *if and only if*

$$\forall k \leq n, X_k \leq \frac{1}{m} \text{ or equivalently } M_n = Max(X_k)_{k \leq n} \leq \frac{1}{m}.$$

Since the right end point d of the common distribution F of the variables X_k is positive, we deduce that the insurance is perfect along any period $[0,T]$ if and only if m is smaller than $\frac{1}{d}$.

For example, if the maximal drop is equal to 20%, then $d = 0,2$. Thus m must be smaller than 5.

2.3 Quantile conditions

The previous condition, which is rather strong, can be modified if a quantile hedging approach is adopted, like for the Value-at-Risk (see Föllmer and Leukert (1999) for recent application of this notion in financial modelling) : for example, an auxiliary floor can be chosen above the initial floor and the new condition is to guarantee that the portfolio value will be always above this new floor at a given probability $1-\epsilon$. This gives the following relation for a period $[0,T]$:

$$\mathbb{P}[C_t \geq 0, \forall t \in [0,T]] \geq 1 - \epsilon,$$

or, equivalently, if M_T denotes the maximum of the X_k for times t_k in $[0,T]$:

$$\mathbb{P}[\forall t_k \in [0,T], X_k \leq \frac{1}{m}] = \mathbb{P}[M_T \leq \frac{1}{m}] \geq 1 - \epsilon.$$

Note that, since m is nonnegative, the condition $X_k \leq \frac{1}{m}$ is equivalent to $X_k \mathbb{I}_{0 \leq X_k} \leq \frac{1}{m}$.

Then we can detail the quantile hedging condition. For this, introduce the function $F_{M_T}^{-1}$ defined as the inverse of the distribution function F which is assumed to be strictly increasing. Then, we get (see Prigent (2001)) :

Proposition 2 *For small r_k,we have approximately the following condition:*

$$m \leq \frac{1}{F_{M_T}^{-1}\left((1-\epsilon)\right)}.$$

Additionally, if the sequence $(X_k)_k$ is i.i.d. with common cdf F_{M_T}, then we have:

$$m \leq \frac{1}{F^{-1}\left((1-\epsilon)^{\frac{1}{T}}\right)}.$$

This condition gives an upper limit on the multiple m which is obviously higher than the standard limit $\frac{1}{d}$.

3 CPPI with a conditional multiple

3.1 Conditional multiple

The simplicity and flexibility of the CPPI method allows the introduction of several extensions:

- First, we can introduce a stochastic floor, in particular to keep past profits from rises in the stock market. For example, Estep and Kritzman (1988) have introduced the Time Invariant Portfolio Protection (TIPP). This method is based on the following guarantee condition:

$$V_t \geq k \times \max(P_t, \sup_{s \leq t} V_s),$$

  where k is an exogenous parameter which lies between 0 and 1. In this case, the investor does not want to lose more than a given percentage of the maximum of his past portfolio values. This strategy is of CPPI type but with a stochastic floor, given by $k \times \max(P_t, \sup_{s \leq t} V_s)$.

- Second, the multiple can be no longer fixed but determined from market and portfolio dynamics:

  For instance, Prigent (2001) introduces a more general exposure function $e(t, X)$ defined on $[0, T] \times \mathbb{R}^+$, positive and continuous.

  This new exposure e has the following form:

$$e_t = e(t, C_t).$$

  Then, conditions on $e(t, X)$ must be imposed such that the cushion always is positive.

  1) If the cushion is nil, then the exposure must be equal to 0:

$$e(t, 0) = 0.$$

2) If the relative sizes of jumps $\frac{\Delta S}{S}$ have a lower bound d (negative), then, for any (t, X), we must have:

$$e(t, X) \leq -\frac{1}{d} X.$$

Another possibility is to chose the multiple according to asset values, turbulence, and market volatility, as proposed by Hamidi *et al.* (2006). In particular, they use a regression quantile method, as introduced by Engel and Manganelli (2004).

In what follows, we examine the problem of the determination of a conditional multiple, using a quantile condition. To illustrate our approach, we assume that the risky asset logreturn follows an Arch type model.

3.2 Quantile Hedging and conditional multiple

3.2.1 Quantile condition

The quantile condition is defined by:

$$\mathbb{P}[\forall t, C_t > 0] \geq 1 - \epsilon. \tag{1}$$

This condition can be deduced for sufficient assumptions. For instance, we have the following result:

Proposition 3 *Consider the filtration $(\mathcal{F}_{t_k})_k$ generated by the observations of the arithmetical returns $(-X_k)_k$.*

Introduce the assumption ($A1$):

$$\forall k, \mathbb{P}[C_{t_k} > 0 \left| C_{t_1} > 0, ..., C_{t_{k-1}} > 0\right] \geq (1 - \epsilon)^{1/T}.$$

Consider also the assumption ($A2$):

$$\forall k, \mathbb{P}^{\mathcal{F}_{t_{k-1}}}[C_{t_k} > 0] \geq (1 - \epsilon)^{1/T}.$$

Finally, introduce the condition ($A3$):

$$\forall k, \mathbb{P}^{\mathcal{G}_{t_{k-1}}}[C_{t_k} > 0] \geq (1 - \epsilon)^{1/T},$$

where $\mathcal{G}_{t_{k-1}}$ is the $\sigma-$algebra generated by $\mathcal{F}_{t_{k-1}}$ and the random event $\left\{C_{t_{k-1}} > 0\right\}$.

Each of these three assumptions implies Property (1).

Proof. Suppose ($A1$) is satisfied. Then, using the equality:

$$\begin{aligned} \mathbb{P}[C_{t_1} &> 0, ...C_T > 0] = \mathbb{P}[C_{t_1} > 0] \times \mathbb{P}[C_{t_2} > 0 \,|C_{t_1} > 0] \times ... \\ ... \times \mathbb{P}[C_T &> 0 \,|C_1 > 0, ...C_{T-1} > 0], \end{aligned}$$

we deduce the result.

Suppose now that $(A2)$ is satisfied.
We use the following lemma: for any random variables X and Y,

$$\mathbb{P}^{Y}[X < 0] \leq a \Rightarrow \mathbb{P}^{Y \in B}[X < 0] \leq a, \forall B \subset \mathbb{R}^{k-1}.$$

Let:

$$X = C_{t_k} \text{ and } Y = (X_{t_1}, ...X_{t_{k-1}}).$$

Introduce the set B defined by:

$$B = \left\{C_{t_1} > 0, ...C_{t_{k-1}} > 0\right\}.$$

Then, we have:

$$\forall k, \mathbb{P}^{C_{t_1} > 0,..C_{t_{k-1}} > 0}[C_{t_k} > 0] \geq (1 - \epsilon)^{1/T};$$

from which, we deduce that assumption (A_1) is satisfied, which implies Property
(1). ■

In what follows, we focus on Conditions (A_2): $\mathbb{P}^{\mathcal{F}_{t_{k-1}}}[C_{t_k} > 0] \geq (1 - \epsilon)^{1/T}$.
Recall that the cushion value is determined from the following relation[2]

$$C_{t_k} = C_{t_{k-1}} \times (1 - m_{t_{k-1}} \times X_{t_k}) > 0.$$

The guarantee condition at any time t_k is: $C_{t_k} > 0$.
Case 1: $(C_{t_{k-1}} > 0)$

$$C_{t_k} > 0 \Longleftrightarrow 1 - m_{t_{k-1}} \times X_{t_k} > 0.$$

Case 2: $(C_{t_{k-1}} < 0)$

$$C_{t_k} > 0 \Longleftrightarrow 1 - m_{t_{k-1}} \times X_{t_k} < 0.$$

As it can be seen, at any time t_{k-1}, we have to choose a value of the multiple $m_{t_{k-1}}$ such that, at time t_k, we still have $C_{t_k} > 0$.
Consequently, we can search $m_{t_{k-1}}$ of the following type:

$$m_{tk-1} = g_{t_{k-1}} \times \mathbb{I}_{C_{tk-1} > 0} + h_{t_{k-1}} \times \mathbb{I}_{C_{tk-1} < 0},$$

where both are random variables which are $\mathcal{F}_{t_{k-1}}$−measurable.
Condition $(A3)$ implies condition $(A1)$. Thus, it also implies Property (1).

[2] Since we consider small time intervals (for example, daily rebalancing), we set $r_k = 0$.

3.2.2 Determination of the multiple

In what follows, we search for explicit forms of the random variables $g_{t_{k-1}}$ and $h_{t_{k-1}}$. We assume that the risky asset logreturn Y follows a Garch(p,q) model. As it well-known, this kind of dynamics is quite suitable to describe asset fluctuations in a discrete-time setting. The ARCH (*Autoregressive Conditionally Heteroscedastic*) models, introduced by Engle (1982), are specific non-linear time series models. They can describe quite exhaustive set of the underlying dynamics. They have been largely applied on macroeconomics and statistical theory.

The GARCH is defined in the following way. Consider the logreturn Y:

$$Y_{t_k} = \ln\left(\frac{S_{t_k}}{S_{t_{k-1}}}\right) \Rightarrow \frac{S_{t_k} - S_{t_{k-1}}}{S_{t_{k-1}}} = \exp(Y_k) - 1.$$

Consider the systems of auto regressive equations:

$$\begin{cases} Y_k = \alpha_0 + \sum_{i=1}^{p} \alpha_i \times Y_{t_{k-i}} + \sigma_k \times \epsilon_k, \\ \Lambda(\sigma_k) = \beta + C_0(\epsilon_{k-1}) + C_1(\epsilon_{k-1}) \times \Lambda(\sigma_{k-1}), \end{cases}$$

where σ_k denotes the volatility, the sequence $(\epsilon_k)_k$ is i.i.d with common pdf $f > 0$ and Λ, $C_0(.)$, and $C_1(.)$ are deterministic functions. In particular, we assume that the function $\Lambda : \mathbb{R}^+ \to \mathbb{R}$ is strictly increasing.

The information delivered by the observation of risk asset returns is generated by the $(\epsilon_{t_1}, ...\epsilon_{t_{k-1}})_k$.

We have:

$$\mathcal{F}_{t_{k-1}} = \sigma - \text{algebra } (\epsilon_{t_1}, ...\epsilon_{t_{k-1}}).$$

Thus the random variables $g_{t_{k-1}}$ and $h_{t_{k-1}}$ are deterministic functions of $(\epsilon_{t_1}, ...\epsilon_{t_{k-1}})$.

Therefore, we have to search for a multiple m_{t_k} which has the following form:

$$m_{tk-1} = g(t_{k-1}, Y_{t_1}, ..., Y_{t_{k-1}}, \sigma_{t_1}, ...\sigma_{t_k}) \times \mathbb{I}_{C_{tk-1}>0}$$

$$+h(t_{k-1}, Y_{t_1}, ..., Y_{t_{k-1}}, \sigma_{t_1}, ...\sigma_{t_k}) \times \mathbb{I}_{C_{tk-1}<0}.$$

Remark 4 *Note also that the random variables $Y_{t_{k-1}}$ are deterministic functions $(\epsilon_{t_1}, ...\epsilon_{t_{k-1}})$. But, for a better financial interpretation of the multiple, we explicitly introduce functions of $(Y_{t_1}, ..., Y_{t_{k-1}})$ itself.*

Indeed, we examine how the conditional multiple depends on both the volatility levels on the past logreturns. These two kinds of variables are here the state variables.

In order to determine the conditional multiple, two cases have to be distinguished[3]:

- The cushion at time $_{t_{k-1}}$ is non negative: $C_{t_{k-1}} > 0$.
- The cushion at time $_{t_{k-1}}$ is non positive: $C_{t_{k-1}} < 0$.

[3] Since we assume that the pdf of the random variables ϵ is strictly positive, the probability of the event $C_{t_{k-1}} = 0$ is null.

Case 1: $C_{t_{k-1}} > 0$ The quantile condition $(A2)$ is the following:

$$\mathbb{P}^{\mathcal{F}_{t_{k-1}}}[1 + m_{t_{k-1}} \times \frac{\Delta S_{t_k}}{S_{t_{k-1}}} > 0] \geq (1-\epsilon)^{1/T},$$

which is equivalent to:

$$\mathbb{P}^{\mathcal{F}_{t_{k-1}}}[m_{t_{k-1}}(\exp(Y_{t_k}) - 1) > -1] \geq (1-\epsilon)^{1/T}$$

At time t_k, we have two cases:

$$\begin{cases} \exp(Y_{t_k}) < 1 : S_{t_k} < S_{t_{k-1}} \ (S \text{ decreases}), \\ \exp(Y_{t_k}) > 1 : S_{t_k} > S_{t_{k-1}} \ (S \text{ increases}). \end{cases}$$

We assume that the multiple $m_{t_{k-1}}$ is non negative (it is the standard assumption. It corresponds to a long position on the risky asset).

Then, we deduce:

$$\mathbb{P}^{\mathcal{F}_{t_{k-1}}}[m_{t_{k-1}}(\exp(Y_k) - 1) > -1) =$$

$$\mathbb{P}^{\mathcal{F}_{t_{k-1}}}[m_{t_{k-1}}(\exp(Y_k) - 1) > -1 \cap (\exp(Y_k) - 1) > 0] +$$

$$\mathbb{P}^{\mathcal{F}_{t_{k-1}}}[m_{t_{k-1}}(\exp(Y_k) - 1) > -1 \cap (\exp(Y_k) - 1) < 0]$$

$$= \mathbb{P}^{\mathcal{F}_{t_{k-1}}}[\exp(Y_k) - 1 > 0] + \mathbb{P}^{\mathcal{F}_{t_{k-1}}}[m_{t_{k-1}}(\exp(Y_k) - 1) > -1 \cap (\exp(Y_{t_k}) - 1) < 0]$$

Since we have:

$$m_{t_{k-1}}(\exp(Y_k) - 1) \geq -1 \Rightarrow Y_{t_k} \succ \ln(1 - \frac{1}{m_{k-1}}),$$

then

$$\mathbb{P}^{\mathcal{F}_{t_{k-1}}}[m_{t_{k-1}}(\exp(Y_k) - 1) > -1) = \mathbb{P}^{\mathcal{F}_{t_{k-1}}}[Y_{t_k} > 0] + \mathbb{P}^{\mathcal{F}_{t_{k-1}}}[0 > Y_{t_k} > \ln(1 - \frac{1}{m_{k-1}})],$$

$$= \mathbb{P}^{\mathcal{F}_{t_{k-1}}}[Y_{t_k} > \ln(1 - \frac{1}{m_{t_{k-1}}})] = 1 - \underbrace{F^{\mathcal{F}_{k-1}}[\ln(1 - \frac{1}{m_{k-1}})]}.$$

Therefore, the quantile condition

$$\mathbb{P}^{\mathcal{F}_{tk-1}, C_{tk-1} > 0}[C_{t_k} > 0] \geq (1-\epsilon)^{1/T},$$

is equivalent to:

$$1 - F^{F_{k-1}}[\ln(1 - \frac{1}{m_{t_{k-1}}})] \geq (1-\epsilon)^{1/T},$$

and also to:

$$\ln(1 - \frac{1}{m_{t_{k-1}}}) \leq (F^{F_{k-1}})^{-1}\left([1 - (1-\epsilon)^{1/T}]\right)$$

and finally, we have:

1-1) If $\exp[(F^{F_{t_{k-1}}})^{-1} \times [1 - (1-\epsilon)^{1/T}]] > 1$, since $m_{t_{k-1}}$ is assumed to be non negative, the quantile condition is already satisfied.

1-2) If $\exp[(F^{F_{t_{k-1}}})^{-1} \times [1-(1-\epsilon)^{1/T}]] < 1$, then $m_{t_{k-1}}$must satisfy the following constraint:

$$m_{t_{k-1}} \leq \frac{1}{1-\exp[(F^{F_{t_{k-1}}})^{-1}\left([1-(1-\epsilon)^{1/T}]\right)]]}.$$

Let us examine the two previous subcases. They are based on the sign of the term:

$$(F^{F_{t_{k-1}}})^{-1}\left([1-(1-\epsilon)^{1/T}]\right).$$

Consequently, we must study the conditional cdf $F^{F_{t_{k-1}}}$. Recall that we assume (GARCH(p,q) model):

$$\left\{ \begin{array}{c} Y_k = \alpha_0 + \sum_{i=1}^{P} \alpha_i \times Y_{t_{k-i}} + \sigma_k \times \epsilon_k, \\ \Lambda(\sigma_k) = \beta + C_0(\epsilon_{k-1}) + C_1(\epsilon_{k-1}) \times \Lambda(\sigma_{k-1}). \end{array} \right.$$

Lemma 5 *For any real numbers a and b ($b>0$) and for any random variable ϵ with pdf f_ϵ, we have:*

$$\mathbb{P}[a+b\epsilon \leq x] = \mathbb{P}[\epsilon < \frac{x-a}{b}] = \int_{-\infty}^{\frac{x-a}{b}} f_\epsilon(u)du.$$

Thus the pdf of $a+b\epsilon$ is given by:

$$f(x) = \frac{1}{b} \times f_\epsilon(\frac{x-a}{b}).$$

We apply this lemma by setting:

$$\left\{ \begin{array}{c} a+b\epsilon = Y_{t_k}, \\ \epsilon = \epsilon_{t_k}, \\ a = \alpha_0 + \sum_{i=1}^{P} \alpha_i \times Y_{t_{k-i}}, \\ b = \sigma_k > 0. \end{array} \right.$$

We get:

$$f_{Y_k}^{F_{k-1}}(y) = \frac{1}{\sigma_{t_k}} \times f_\epsilon\left(\frac{y-(\alpha+\sum_{i=1}^{P} Y_{t_{k-i}})}{\sigma_{t_k}}\right)$$

$$F_{Y_k}^{F_{k-1}}(y) = F\left(\frac{y-(\alpha+\sum_{i=1}^{P} Y_{t_{k-i}}}{\sigma_{t_k}}\right),$$

where F denotes the common cdf of the random variables ϵ_k.

Thus, we deduce:

- The condition in subcase (1-1) is satisfied if and only if

$$\alpha_0 + \sum_{i=1}^{P} \alpha_i \times Y_{t_{k-i}} + \sigma_k \times F^{-1}(1-(1-\epsilon)^{1/T}) > 0.$$

- The condition in subcase (1-2) is satisfied if and only if

$$\alpha_0 + \sum_{i=1}^{P} \alpha_i \times Y_{t_{k-i}} + \sigma_k \times F^{-1}(1-(1-\epsilon)^{1/T}) < 0,$$

  in which case, the conditional multiple must satisfy:

$$m_{tk-1} \leq \frac{1}{1-\exp[(\alpha_0 + \sum_{i=1}^{P} \alpha_i \times Y_{t_{k-i}}) + \sigma_k \times F^{-1}(1-(1-\epsilon)^{1/T})]}$$

Remark 6 *The previous result shows that, if at time t_{k-1}, the auto regressive terms $\alpha_0 + \sum_{i=1}^{P} \alpha_i \times Y_{t_{k-i}}$ and the conditional volatility σ_k are sufficiently high, then the logreturn Y_{t_k} has a sufficiently high probability to be positive and thus, there is no constraint on the multiple at time t_{k-1}*[4].

Case 2: $C_{t_{k-1}} < 0$ **and** $m_{t_{k-1}} > 0$ As previously, the quantile condition is given by:

$$\mathbb{P}^{\mathcal{F}_{t_{k-1}}}[1 + m_{t_{k-1}} \times \frac{\Delta S_{t_k}}{S_{t_{k-1}}} < 0] \geq (1-\epsilon)^{1/T},$$

which is equivalent to:

$$\mathbb{P}^{\mathcal{F}_{t_{k-1}}}[m_{t_{k-1}}(\exp(Y_{t_k}) - 1) < -1] \geq (1-\epsilon)^{1/T}$$

At time t_k, we have two cases:

$$\begin{cases} \exp(Y_{t_k}) < 1 : S_{t_k} < S_{t_{k-1}} \ (S \text{ decreases}), \\ \exp(Y_{t_k}) > 1 : S_{t_k} > S_{t_{k-1}} \ (S \text{ increases}). \end{cases}$$

We assume that the multiple $m_{t_{k-1}}$ is non negative.

$$\begin{aligned} \mathbb{P}^{\mathcal{F}_{t_{k-1}}}[m_{t_{k-1}}(\exp(Y_{t_k}) - 1) &< -1] = \\ \mathbb{P}^{\mathcal{F}_{t_{k-1}}}[m_{t_{k-1}}(\exp(Y_{t_k}) - 1) &< -1 \cap (\exp(Y_{t_k}) - 1) > 0] \\ +\mathbb{P}^{\mathcal{F}_{t_{k-1}}}[m_{t_{k-1}}(\exp(Y_{t_k}) - 1) &< -1 \cap (\exp(Y_{t_k}) - 1) < 0]. \end{aligned}$$

Therefore, we have:

$$\begin{aligned} \mathbb{P}^{\mathcal{F}_{t_{k-1}}}[m_{t_{k-1}}(\exp(Y_{t_k}) - 1) &< -1] = \\ \mathbb{P}^{\mathcal{F}_{t_{k-1}}}[m_{t_{k-1}}(\exp(Y_{t_k}) - 1) &< -1 \cap (\exp(Y_{t_k}) - 1) < 0] = \\ \mathbb{P}^{\mathcal{F}_{t_{k-1}}}[m_{t_{k-1}}(\exp(Y_{t_k}) - 1) &< -1] \end{aligned}$$

[4] Note that $F^{-1}(1-(1-\epsilon)^{1/T})$ is a particular value of the random variable ϵ_k (since it is the quantile at the level $(1-(1-\epsilon)^{1/T})$).

Consequently, we get the equivalence between the two following conditions:

$$\mathbb{P}^{\mathcal{F}_{t_{k-1}}}[1 + m_{t_{k-1}} \times \frac{\Delta S_{t_k}}{S_{t_{k-1}}} < 0] \geq (1-\epsilon)^{1/T},$$

$$\mathbb{P}^{\mathcal{F}_{t_{k-1}}}[Y_{t_k} \leq \ln(1 - \frac{1}{m_{k-1}})] \geq (1-\epsilon)^{1/T}.$$

Thus, the quantile condition is also equivalent to:

$$F^{F\ t_{k-1}}[\ln(1 - \frac{1}{m_{k-1}})] \geq (1-\epsilon)^{1/T},$$

$$\ln(1 - \frac{1}{m_{k-1}}) \geq (F^{F\ t_{k-1}})^{-1} \left[(1-\epsilon)^{1/T}\right],$$

$$(1 - \frac{1}{m_{k-1}}) \geq \exp((F^{F\ t_{k-1}})^{-1} \left[(1-\epsilon)^{1/T}\right]).$$

2-1) If $\exp[(F^{F\ t_{k-1}})^{-1}[(1-\epsilon)^{1/T}]] > 1$, there is no positive solution for $m_{t_{k-1}}$.

2-2) If $\exp[(F^{F\ t_{k-1}})^{-1}[(1-\epsilon)^{1/T}]] < 1$, then $m_{t_{k-1}}$must satisfy the following constraint:

$$m_{t_{k-1}} \geq \frac{1}{1 - \exp[(F^{F\ t_{k-1}})^{-1}\left([(1-\epsilon)^{1/T}]\right)]]}.$$

Thus, we deduce:

- The condition in subcase (2-1) is satisfied if and only if

$$\alpha_0 + \sum_{i=1}^{P} \alpha_i \times Y_{t_{k-i}} + \sigma_k \times F^{-1}((1-\epsilon)^{1/T}) > 0.$$

- The condition in subcase (2-2) is satisfied if and only if

$$\alpha_0 + \sum_{i=1}^{P} \alpha_i \times Y_{t_{k-i}} + \sigma_k \times F^{-1}((1-\epsilon)^{1/T}) < 0,$$

  in which case, the conditional multiple must satisfy:

$$m_{tk-1} \geq \frac{1}{1 - \exp[(\alpha_0 + \sum_{i=1}^{P} \alpha_i \times Y_{t_{k-i}}) + \sigma_k \times F^{-1}((1-\epsilon)^{1/T})]}.$$

Case 3: $C_{t_{k-1}} < 0$ **and** $m_{t_{k-1}} < 0$ The quantile condition is:

$$\mathbb{P}^{\mathcal{F}_{t_{k-1}}}[1 + m_{t_{k-1}} \times \frac{\Delta S_{t_k}}{S_{t_{k-1}}} < 0] \geq (1-\epsilon)^{1/T},$$

which is equivalent to:

$$\mathbb{P}^{\mathcal{F}_{t_{k-1}}}[m_{t_{k-1}}(\exp(Y_{t_k}) - 1) < -1] \geq (1-\epsilon)^{1/T}$$

We have:

$$\mathbb{P}^{\mathcal{F}_{t_{k-1}}}[m_{t_{k-1}}(\exp(Y_{t_k}) - 1) < -1] =$$

$$\begin{aligned} \mathbb{P}^{\mathcal{F}_{t_{k-1}}}[m_{t_{k-1}}(\exp(Y_{tk}) - 1) &< -1 \, \cap \, (\exp(Y_{tk}) - 1) > 0] \\ +\mathbb{P}^{\mathcal{F}_{t_{k-1}}}[m_{t_{k-1}}(\exp(Y_{tk}) - 1) &< -1 \, \cap \, (\exp(Y_{tk}) - 1) < 0] \end{aligned}$$

$$\begin{aligned} &= \mathbb{P}^{\mathcal{F}_{t_{k-1}}}[m_{tk-1}(\exp(Y_{t_k}) - 1) < -1] \\ &= \mathbb{P}^{\mathcal{F}_{t_{k-1}}}[\exp(Y_{t_k}) > 1 - \frac{1}{m_{t_{k-1}}}] \end{aligned}$$

Therefore the quantile condition is equivalent to:

$$1 - F^{F\ tk-1}[\ln(1 - \frac{1}{m_{t_{k-1}}})] \geq (1-\epsilon)^{1/T},$$

which is also:

$$(1 - \frac{1}{m_{k-1}}) \leq \exp((F^{F\ t_{k-1}})^{-1}\left[1 - (1-\epsilon)^{1/T}\right]).$$

3-1) If $\exp[(F^{F\ t_{k-1}})^{-1}[1-(1-\epsilon)^{1/T}]] > 1$, then $m_{t_{k-1}}$must satisfy the following constraint:

$$m_{t_{k-1}} \leq \frac{1}{1 - \exp[(F^{F\ tk-1})^{-1}\left([1-(1-\epsilon)^{1/T}]\right)]]}.$$

3-2) If $\exp[(F^{F\ t_{k-1}})^{-1}[1-(1-\epsilon)^{1/T}]] < 1$, there is no negative solution for $m_{t_{k-1}}$.

Thus, we deduce:

- The condition in subcase (3-1) is satisfied if and only if

$$\alpha_0 + \sum_{i=1}^{P} \alpha_i \times Y_{t_{k-i}} + \sigma_k \times F^{-1}\left([1-(1-\epsilon)^{1/T}]\right) > 0.$$

- The condition in subcase (3-2) is satisfied if and only if

$$\alpha_0 + \sum_{i=1}^{P} \alpha_i \times Y_{t_{k-i}} + \sigma_k \times F^{-1}\left([1-(1-\epsilon)^{1/T}]\right) < 0,$$

in which case, the conditional multiple must satisfy:

$$m_{t_{k-1}} \leq \frac{1}{1-\exp[(\alpha_0 + \sum_{i=1}^{P} \alpha_i \times Y_{t_{k-i}}) + \sigma_k \times F^{-1}\left([1-(1-\epsilon)^{1/T}]\right)]}.$$

To summarize, we get the following result.

Denote $Z_{t_{k-1}} = \alpha_0 + \sum_{i=1}^{P} \alpha_i \times Y_{t_{k-i}} + \sigma_k \times F^{-1}(1-(1-\epsilon)^{1/T})$ and $W_{t_{k-1}} = \alpha_0 + \sum_{i=1}^{P} \alpha_i \times Y_{t_{k-i}} + \sigma_k \times F^{-1}((1-\epsilon)^{1/T})$.

Proposition 7 *The quantile condition (A2) at time t_{k-1} can be satisfied as soon as the cushion value at time t_{k-1} is non negative. In this case, if $Z_{t_{k-1}} > 0$, then the multiple can take any positive value, and, if $Z_{t_{k-1}} < 0$, then the conditional multiple must satisfy:*

$$m_{t_{k-1}} \leq \frac{1}{1-\exp[Z_{t_{k-1}}]}.$$

If the cushion value at time t_{k-1} is non positive, then:

- If $W_{t_{k-1}} < 0$, then there exist positive solutions $m_{t_{k-1}}$,which must satisfy the following constraint:

$$m_{t_{k-1}} \geq \frac{1}{1-\exp[W_{t_{k-1}}]]}.$$

- If $Z_{t_{k-1}} > 0$, then there exist negative solutions $m_{t_{k-1}}$, which must satisfy the following constraint:

$$m_{t_{k-1}} \leq \frac{1}{1-\exp[Z_{t_{k-1}}]]}.$$

- If $W_{t_{k-1}} > 0$ or $Y_{t_{k-1}} > 0$, there exist respectively no positive solution and no negative solution.

Corollary 8 *The quantile condition (A3) at time t_{k-1} can always be satisfied. If $Z_{t_{k-1}} > 0$, then the multiple can take any positive value, and, if $Z_{t_{k-1}} < 0$, then the conditional multiple must satisfy:*

$$m_{t_{k-1}} \leq \frac{1}{1-\exp[Z_{t_{k-1}}]}.$$

Remark 9 *When the cushion is positive at time t_{k-1}, the choice of the multiple is very flexible. Thus, within the quantile condition at time t_{k-1}, we can* *add some other conditions on the multiple to better benefit from market conditions. When the cushion is negative (which happens with a small probability), the quantile condition generally cannot be satisfied, except for small values of $(1-\epsilon)$. But, in this case, this is not a true insurance condition. Therefore, a possible strategy is to adopt the previous condition when the cushion is positive and to invest the whole portfolio value on the riskless asset, as soon as the cushion is negative.*

4 Simulations

4.1 First case: i.i.d. returns and fixed multiple

We can determine the value of the multiple according to assumptions on logreturn process.

For example, suppose that S_t follows a geometric Brownian motion:

$$S_{t+1} = S_t \times \exp((\mu - \frac{1}{2} \times \sigma^2)t + \sigma \times \sqrt{t} \times Z_t),$$

where, in particular, Z_t has a standard Gaussian distribution $N(0,1)$. Denote by N its cdf.

In that case, the previous conditions are made on $Z_{t_{k-1}}$ and $W_{t_{k-1}}$ which now are constant:

$$Z_{t_{k-1}} = (\mu - \frac{1}{2} \times \sigma^2)\Delta t + \sigma\sqrt{\Delta t} \times N^{-1}(1-(1-\epsilon)^{1/T}),$$

and

$$W_{t_{k-1}} = (\mu - \frac{1}{2} \times \sigma^2)\Delta t + \sigma\sqrt{\Delta t} \times N^{-1}((1-\epsilon)^{1/T}).$$

- Consider for instance the following parameter values:

$$\begin{cases} \mu = 0.1, \\ \sigma = 0.3. \end{cases}$$

Proposition 10 *Since the fixed multiple m is positive, then the probability that there exists a time t such that the cushion C_t is negative is given by:*

$$1 - \mathbb{P}\left[\forall t, C_t > 0\right] = 1 - \left(1 - F\left(\ln\left[1 - \frac{1}{m}\right]\right)\right)^T,$$

where

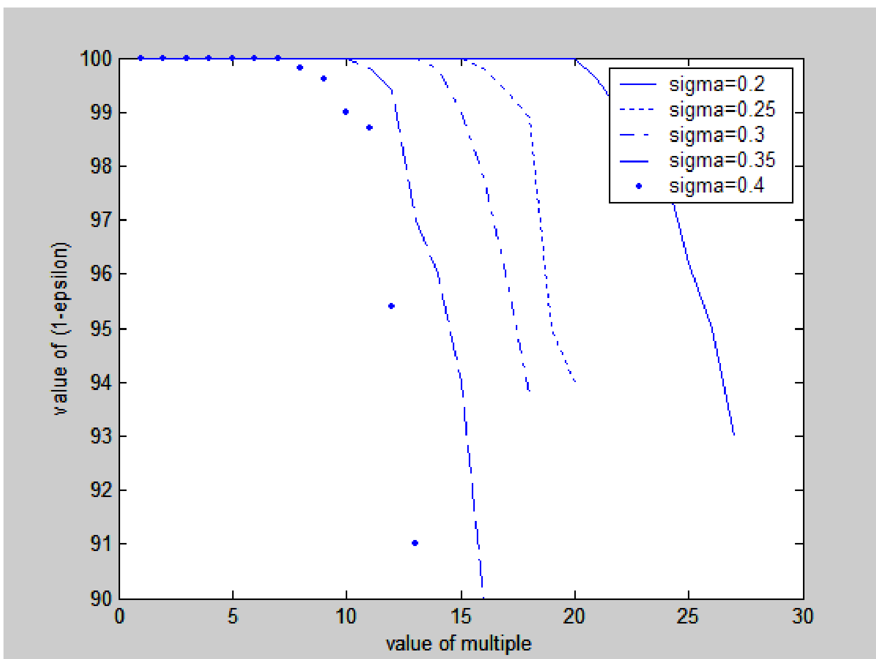
$$F(x) = N\left[\frac{x - (\mu - \frac{1}{2} \times \sigma^2)\Delta t}{\sigma\sqrt{\Delta t}}\right].$$

In what follows, we determine the correspondence between the level of probability $(1-\epsilon)$ and the multiple m.

For instance, we fix the value of the instantaneous rate of return: $\mu = 5\%$ and consider a varying volatility.

m ($\sigma = 0,2$)	19	20	21	22	23	24	25	26
$(1-\epsilon)$ en %	100	99,6	99	98.5	97.8	96.2	95	93
m ($\sigma = 0,25$)	13	14	15	16	17	18	19	20
$(1-\epsilon)$ en %	100	100	100	99.8	99.4	98.9	95	94.5
m ($\sigma = 0,30$)	12	13	14	15	16	17	19	20
$(1-\epsilon)$ en %	100	100	99.8	99	97.8	96	93.8	90.5
m ($\sigma = 0,35$)	9	10	11	12	13	14	15	16
$(1-\epsilon)$ en %	100	100	99.8	99.4	97	96	94	90
m ($\sigma = 0,40$)	7	8	9	10	11	12	13	14
$(1-\epsilon)$ en %	100	99.8	99.6	99	98.7	98	95.4	91

As shown in previous table, the volatility levels determine the multiple values. This is also illustrated by the next figure.



Note that the higher the volatility σ, the smaller the multiple value m for fixed probability level $(1-\varepsilon)$.

For example, if $T = 1$ year, for daily volatilities equal to $\sigma = 0,40 \times \sqrt{1/250}$, the conditional multiple varies between 7 and 14.

Recall that the usual values of the non conditional multiple are in $\{6, 10\}$.

4.2 Dependent returns (ARCH type models)

Consider for instance the Garch(1,1) model with parameter values such as in Nelson (1990).

$$\begin{cases} Y_{t_k} = (\mu - \frac{1}{2}\sigma_*^2)\Delta t + \sigma_{t_k} \times \sqrt{\Delta t} \times \epsilon_{t_k}, \\ \sigma_{tk} = \sigma_{t_{k-1}} + \beta(a - \sigma_{t_{k-1}})\Delta t + \gamma \times \epsilon_{t_k} \times \sqrt{\Delta t}. \end{cases}$$

4.2.1 Convergence of the Garch(1,1) model to an Ornstein-Uhlenbeck process

The continuous-time model is given by:

$$d\sigma_t = \beta(a - \sigma_t)dt + \gamma dW_t,$$

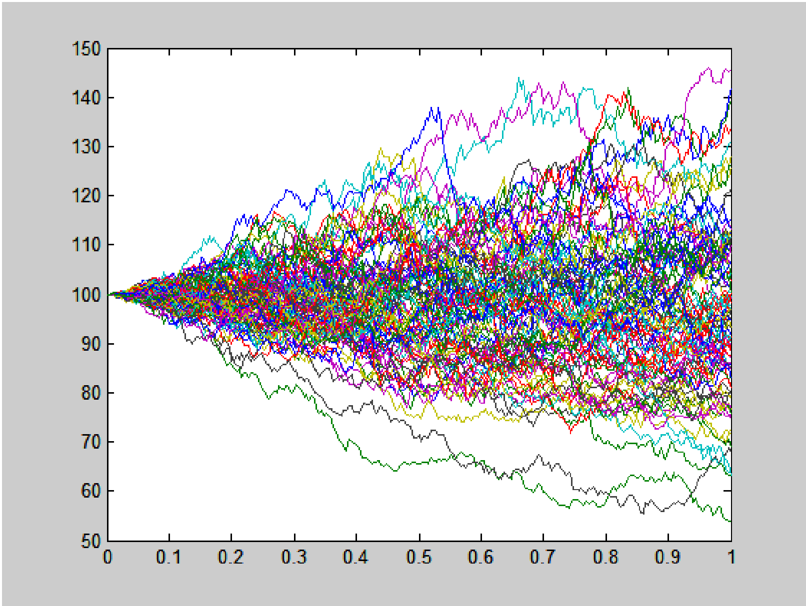
where β is the speed of convergence to the long term value of the volatility a. The parameter γ denotes the volatility of the volatility. We get:

$$\sigma_t = a - a\sigma_0 \exp(-\beta t) + \gamma \int_0^t \exp(\beta(s-t)\ dW_s.$$

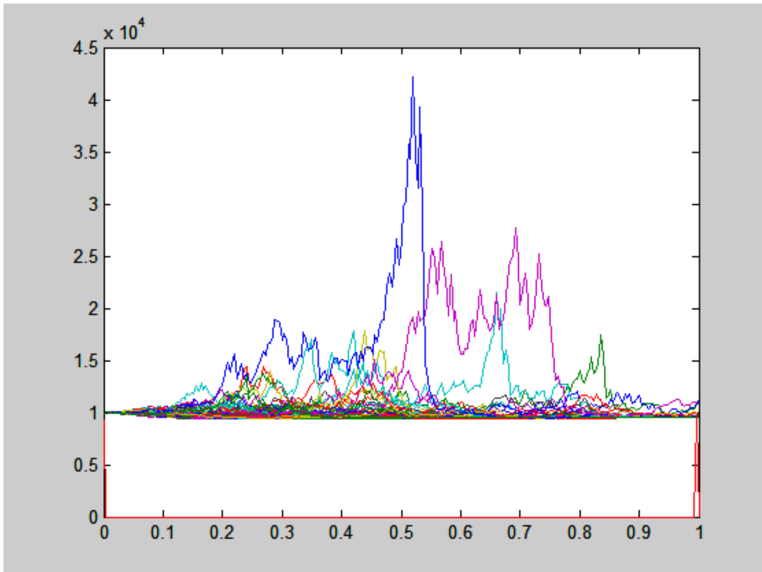
4.2.2 Determination of the conditional multiple from Garch (1,1) model.

Consider the following parameter values: $\sigma = 0.2$, $\mu = 0.07$, $\beta = 5$, $T = 250$, $a = 0.2$

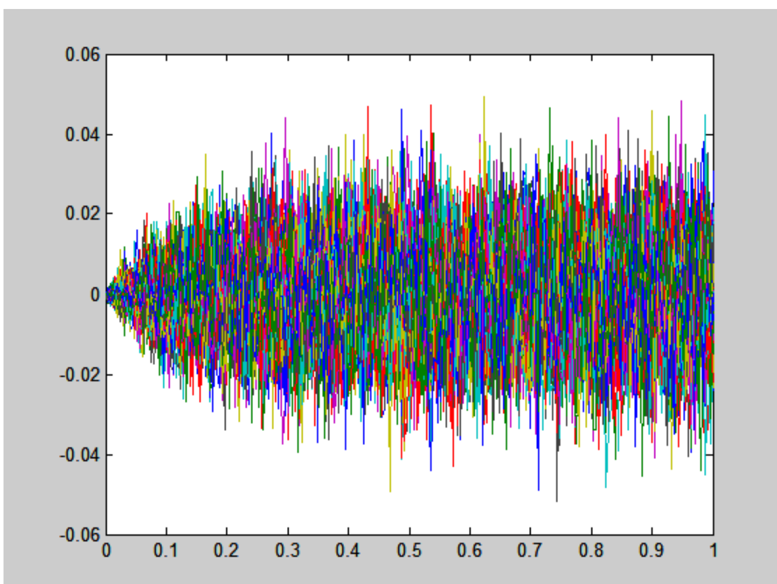
1) Next figure illustrate the paths of the risky asset, under the Garch assumptions.



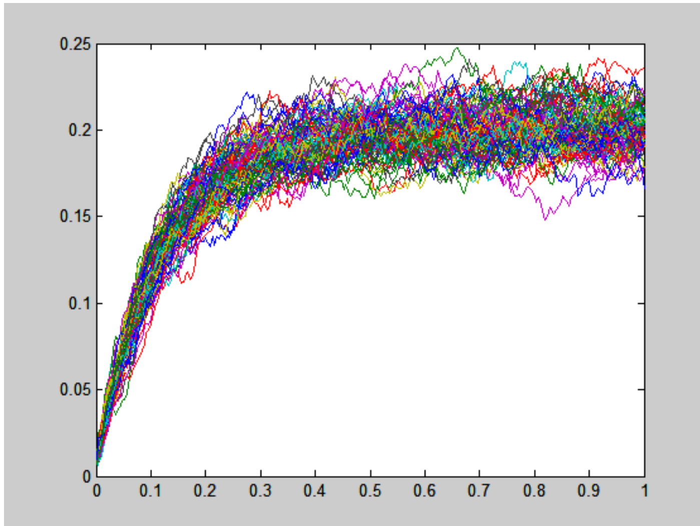
2) Paths of the portfolio value are as follows (recall that we take $V_0 = 10^4$).



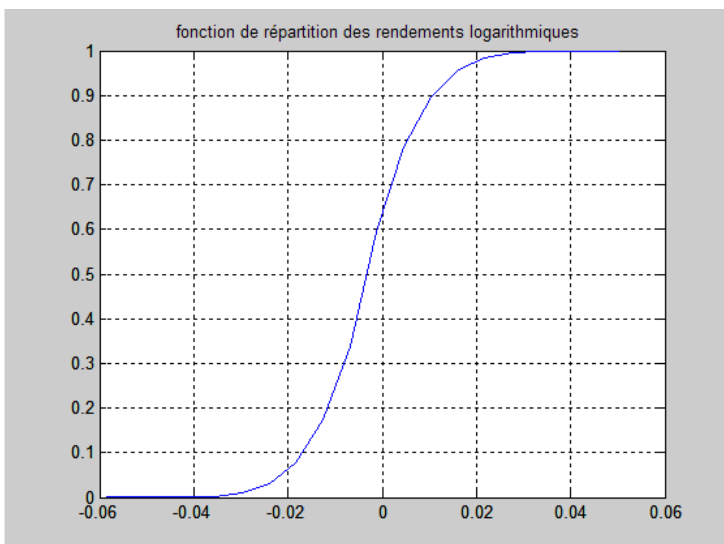
3) We obtain daily logreturns whosevalues are most of the time between -0.02% and 0.02%.



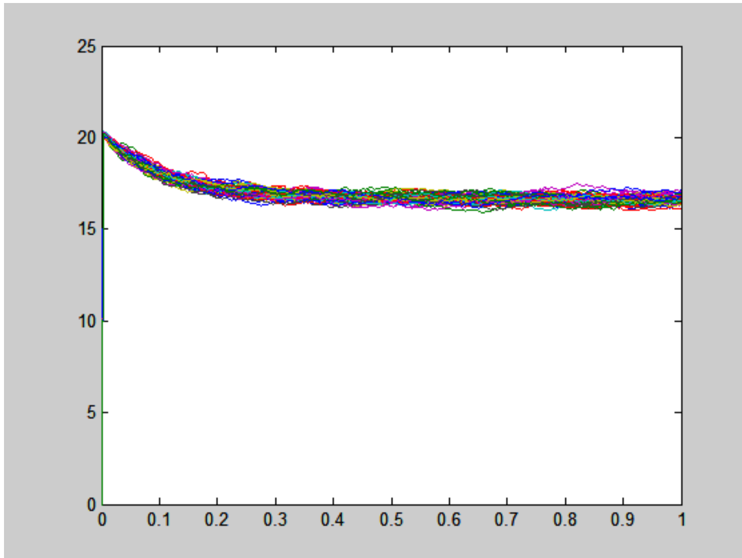
4) The volatility levels are indicated in the following figure. For the parametric specification that we consider, the volatility values converge to the long term value equal to 2%.



5) The cdf of the daily logreturns is gaussian and illustrated below:



6) For the parameter values of this example, the variations of the conditional multiple are shown in next figure.



The previous values of the conditional multiple imply the insurance quantile condition. Since these values are higher than for the standard case, better performances can be observed when the risky asset increases. However, other conditions can be imposed to limit downside risk, while allowing the quantile condition.

5 Conclusion

As shown in this paper, it is possible to choose higher multiples for the CPPI method if quantile hedging is used. Upper bounds can be calculated for each level of probability and according to state variables. This allows the introduction of a conditional multiple. This new multiple can be determined according to the distributions of the risky asset logreturn and volatility. Other conditions can be imposed on this multiple, while the quantile hedging constraint is satisfied. The difference with the standard multiple is significant. Further extensions may allow to better take account of the potential losses, when financial asset prices decrease. For example, criterion such as the Expected Shortfall can be introduced. Other state variables can also be considered, such as exogenous macro economic factors. Finally, the impact of transaction costs can also be examined.

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